# A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves

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The geodesic as well as the geodesic deviation equation for impulsive gravitational waves involve highly singular products of distributions ($\theta\delta$, $\theta^2\delta$, $\delta^2$). A solution concept for these equations based on embedding the distributional metric into the Colombeau algebra of generalized functions is presented. Using a universal regularization procedure we prove existence and uniqueness results and calculate the distributional limits of these solutions explicitly. The obtained limits are regularization independent and display the physically expected behavior.

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## I. INTRODUCTION

Impulsive pp-waves (plane fronted gravitational waves with parallel rays) can be described by a metric of the form [1]

$$ds^2 \,=\, \delta(u)\, f(x,y) du^2 - du\, dv + dx^2 + dy^2, \tag{1}$$

where $(u,v)$ and $(x,y)$ is a pair of null and (transverse) Cartesian coordinates respectively, and $f$ denotes the profile function subject to the field equations. Hence the spacetime is flat everywhere except for the null hypersurface $u=0$, where it has a $\delta$-like pulse modelling a gravitational shock wave. Such geometries arise most naturally as ultrarelativistic limits of boosted black hole spacetimes of the Kerr-Newman family (as shown by various authors [2–4]) and multipole solutions of the Weyl family [5]. Also they play an important role in particle scattering at the Planck scale (see [6] and references therein).

There have been intrinsic descriptions of impulsive pp-waves, viz. by Penrose [1] and by Dray and t'Hooft [7], which essentially consist in glueing together two copies of Minkowski spacetime with a warp across the null hypersurface $u=0$. Penrose also introduced a different coordinate system in which the components of the metric tensor are actually continuous. However, the transformation relating the coordinates used in (1) to these new ones is discontinuous (for the general form of the transformation see [8]) and therefore –strictly speaking– the differential structure of the manifold is changed. In this paper we stick to the original distributional form of the metric, motivated by the fact that physically, i.e. in the ultrarelativistic limit, the spacetime arises that way (cf. the approaches of [9,10]). For recent work on pp-waves using the continuous form of the metric see [11].

We describe the geometry of impulsive pp-waves entirely in the distributional picture using the framework of Colombeau's generalized functions, thereby generalizing previous work [12]. As discussed there in detail, the geodesic as well as the geodesic deviation equation for impulsive pp-waves involve formally ill-defined products of distributions, due to the nonlinearity of the equations and the presence of the Dirac $\delta$-function in the space time metric. However, as was also shown in [12], one can overcome these difficulties using a careful regularization procedure which, while mathematically sound, corresponds to the physical idea of viewing the impulsive wave as the limiting case of a sandwich wave of ever decreasing support but constant (integrated) strength. More precisely, regularizing the $\delta$-distribution by a "model $\delta$-net" (i.e., a net $\rho_\epsilon(x) := \epsilon^{-1}\rho(x\epsilon^{-1})$, where $\rho$ is a smooth function with support contained in the interval $[-1,1]$ satisfying $\int\rho = 1$), it was shown that the solutions to the smoothened equations possess regularization independent weak limits. These distributional "solutions" fit perfectly into the physically expected picture showing that the geometry of impulsive pp-waves can be described consistently using the distributional form of the metric.

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The reliability of the results is guaranteed by making use of regularization techniques instead of introducing "multiplication rules" into Schwartz linear disribution theory (cf. the discussion at the end of Sec. 2 in [12] or [13] for general remarks).
However, the "solutions" obtained by this naive regularization procedure exhibit a mathematically highly unsatisfactory feature. *They do not obey the original distributional equations* (unless, again, one is willing to impose certain "multiplication rules"), as is common to such situations. Hence –strictly speaking– this approach does not provide a reasonable solution concept for the equations under consideration. Such a notion *is* available in the nonlinear theory of generalized functions [14–16] due to J. F. Colombeau, where one has –loosely speaking– a rigorous system of bookkeeping on the regularizing sequences. Recently Hermann and Oberguggenberger [17] (see also [18]) studied systems of singular, nonlinear ODEs in the Colombeau algebra. In this work we are going to use similar techniques to treat the geodesic and geodesic deviation equation for impulsive pp-waves in the Colombeau algebra. Despite the nonlinearities involved in these equations (which in principle could lead to trapping, blow–up or reflection of solutions at the shock, cf. [17]) we are able to prove existence and uniqueness of geodesics crossing the shock hypersurface. We derive the (regularization independent) distributional limits of these solutions, making use of the notion of association (see Sec. II below) in the algebra, thereby significantly generalizing the results of [12]. In particular, the regularization of the $\delta$-like wave profile will no longer be restricted to a "model $\delta$-net" but belong to the largest "reasonable" class (cf. Definition 1 below). Moreover, note that the regularization idependence of the results has the following important physical consequence: in the impulsive limit the geodesics are totally independent of the particular shape of the sandwich wave. Hence the impulsive wave "totally forgets its seed" (cf. also the results in [19]).
Finally, we discuss the case of a nonsmooth wave profile $f$ and give an outlook to current research which allows to fit our previous calculations into a manifestly covariant concept of Colomebeau algebras on manifolds.

## II. MATHEMATICAL FRAMEWORK

A framework that allows consistent treatment of nonlinear operations with distributions and at the same time offers a well-developed theory of (linear and nonlinear) partial differential equations is provided by Colombeau's theory of algebras of generalized functions (cf. e.g. [14–16,20]). To begin with, we give a short description of the algebra we are going to use in the sequel. Let

$$\mathcal{A}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{D}(\mathbb{R}^n)\,:\, \int \varphi(x)\,dx = 1\}$$
$$\mathcal{A}_q(\mathbb{R}^n) = \{\varphi \in \mathcal{A}_0(\mathbb{R}^n)\,:\, \int \varphi(x)x^\alpha\,dx = 0\,,\, 1 \leq |\alpha| \leq q\}\ \ (q \in \mathbb{N})$$

and set (for any $\Omega \subseteq \mathbb{R}^n$ open)

$$\begin{aligned}
\mathcal{E}(\Omega) &= \{R : \mathcal{A}_0(\mathbb{R}^n) \times \Omega \to \mathbb{C}\,:\, x \to R(\varphi, x) \in \mathcal{C}^\infty(\Omega)\,\forall \varphi \in \mathcal{A}_0(\mathbb{R}^n)\}\\
\mathcal{E}_M(\Omega) &= \{u \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists p \in \mathbb{N}_0\ \forall \varphi \in \mathcal{A}_p(\mathbb{R}^n)\\
&\qquad \exists c > 0\ \exists \eta > 0\ \sup_{x\in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{-p}\ (0 < \varepsilon < \eta)\}\\
\mathcal{N}(\Omega) &= \{u \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists p \in \mathbb{N}_0\ \exists \gamma \in \Gamma\ \forall q \geq p\\
&\qquad \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)\ \exists c > 0\ \exists \eta > 0\ \sup_{x\in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q)-p}\ (0 < \varepsilon < \eta)\}\ ,
\end{aligned}$$

where $\Gamma = \{\gamma : \mathbb{N}_0 \to \mathbb{R}_+ : \gamma \text{ strictly increasing}, \lim_{n\to\infty} \gamma(n) = \infty\}$. Derivation $\partial^\alpha$ is carried out with respect to $x$, while the $\varphi$ are treated as parameters. Also, for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$. Note that $\varphi_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$.
Elements of $\mathcal{E}_M(\Omega)$ are called of *moderate growth*. With pointwise operations $\mathcal{E}_M(\Omega)$ is a differential algebra and $\mathcal{N}(\Omega)$ is an ideal in $\mathcal{E}_M(\Omega)$. The quotient algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

is called the *Colombeau algebra* over $\Omega \subseteq \mathbb{R}^n$. Elements of $\mathcal{G}(\Omega)$ will be denoted by $R = \mathrm{cl}[(R(\varphi,\,.\,))_{\varphi\in\mathcal{A}_0}]$ where $(R(\varphi,\,.\,))_{\varphi\in\mathcal{A}_0}$ is an arbitrary representative of $R$ (again emphasizing the fact that the $\varphi$'s are viewed as parameters).
For $\Omega = \mathbb{R}^n$ the map

$$\begin{aligned}
\iota &: \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega)\\
w &\to \mathrm{cl}[(w * \varphi)_{\varphi\in\mathcal{A}_0}]
\end{aligned}$$

(where $*$ denotes convolution) is a linear embedding commuting with partial derivatives and coinciding with the identical embedding $f \to \mathrm{cl}[(f)_{\varphi\in\mathcal{A}_0}]$ on $\mathcal{D}(\mathbb{R}^n)$.
$\mathcal{G}$ is a fine sheaf of differential algebras on $\mathbb{R}^n$ and there is a unique sheaf morphism $\widehat{\iota}: \mathcal{D}' \to \mathcal{G}$ coinciding with $\iota$ on every $\mathcal{E}'(\Omega)$ and rendering $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra of $\mathcal{G}(\Omega)$. From the definitions it is clear that any element of $\mathcal{G}(\Omega)$ is uniquely determined by the values of any representative on $\varphi_\varepsilon$ for $\varphi \in \mathcal{A}_p$ with $p$ arbitrarily large and $\varepsilon$ arbitrarily small (i.e. by its 'germ'), a fact that turns out to be very helpful e.g. in constructing solutions to differential equations in $\mathcal{G}$.
Inserting points into elements of $\mathcal{G}(\mathbb{R}^n)$ gives elements of the ring of generalized numbers $\overline{\mathbb{C}}(n)$, defined as $\overline{\mathbb{C}}(n) = \mathcal{E}(n)/\mathcal{N}(n)$, where

$$\begin{aligned}\mathcal{E}(n) = \{u : \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C} : \exists p \in \mathbb{N}_0\ \forall \varphi \in \mathcal{A}_p(\mathbb{R}^n)\ \exists c > 0\ \exists \eta > 0 \\ |u(\varphi_\varepsilon)| \leq c\varepsilon^{-p}\ (0 < \varepsilon < \eta)\} \\ \mathcal{N}(n) = \{u : \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C} : \exists p \in \mathbb{N}_0\ \exists \gamma \in \Gamma\ \forall q \geq p\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n) \\ \exists c > 0\ \exists \eta > 0\ |u(\varphi_\varepsilon)| \leq c\varepsilon^{\gamma(q)-p}\ (0 < \varepsilon < \eta)\}\end{aligned}$$

Thus elements of $\mathcal{G}(\mathbb{R}^n)$ take values in $\overline{\mathbb{C}}(n)$. Explicit dependence of the ring of constants on $n$ can be avoided by a more refined construction of the sets $\mathcal{A}_q$ in the definition of $\mathcal{G}$ (see [20]). Clearly, $\mathbb{C} \hookrightarrow \overline{\mathbb{C}}$ via the canonical embedding $c \to \mathrm{cl}[(c)_{\varphi\in\mathcal{A}_0}]$.
Componentwise insertion of $R \in \mathcal{G}$ into a smooth function $f$ yields a well defined element $f(R)$ of $\mathcal{G}$ if $f$ is *slowly increasing*, i.e. if all derivatives of $f$ are polynomially bounded. Moreover, if $R$ is *locally bounded,* i.e. if it possesses a representative such that $R(\varphi_\varepsilon, .)$ is bounded uniformly in $\varepsilon$ on compact sets (for $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$, $p$ large) then $f \circ R$ exists for any smooth $f$.
Finally, we mention the notion of *association* in $\mathcal{G}(\Omega)$: $R_1$, $R_2 \in \mathcal{G}(\Omega)$ are called associated to each other ($R_1 \approx R_2$) if there exists some $p \in \mathbb{N}$ such that $R_1(\varphi_\varepsilon, .) - R_2(\varphi_\varepsilon, .) \to 0$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$ for all $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$. In particular, if $R_2 \in \mathcal{D}'(\Omega)$ then $R_2$ is called the macroscopic aspect (or *distributional shadow*) of $R_1$. Equality in $\mathcal{D}'$ is reflected as equality in the sense of association in $\mathcal{G}$, while equality in $\mathcal{G}$ is a stricter concept (for example, all powers of the Heaviside function are distinct in the Colombeau algebra although they are associated with each other).

## III. EXACT SOLUTIONS OF GEODESIC AND GEODESIC DEVIATION EQUATIONS

As in [12] we consider the impulsive pp-wave metric

$$ds^2 \;=\; f(x^i)\,\delta(u)\,du^2 - du\,dv + (dx^i)^2\ , \tag{2}$$

where $f$ is a smooth function of the transverse coordinates $x^i$ ($i = 1, 2$). Our aim is to derive solutions to the corresponding geodesic and geodesic deviation equations in the Colombeau algebra.
The general strategy for solving differential equations in $\mathcal{G}$ is to embed singularities (in our case: $\delta$) into $\mathcal{G}$ which amounts to a regularization and then solve the corresponding regularized equations. In order to obtain general results we are therefore interested in imposing as few restrictions as possible on the regularization of $\delta$. The largest "reasonable" class of smooth[1] regularizations of $\delta$ is given by nets $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ of smooth functions $\rho_\varepsilon$ satisfying:

$$\begin{aligned}&\text{(a)}\quad \mathrm{supp}(\rho_\varepsilon) \to \{0\} \quad (\varepsilon \to 0)\ ,\\ &\text{(b)}\quad \int \rho_\varepsilon(x)\,dx \to 1 \quad (\varepsilon \to 0)\ \ \text{and}\\ &\text{(c)}\quad \exists \eta > 0\ \exists C \geq 0 : \int |\rho_\varepsilon(x)|\,dx \leq C\ \forall \varepsilon \in (0, \eta)\end{aligned}$$

(cf. the definition of *strict delta nets* in [16], ch. 2.7). Obviously any such net converges to $\delta$ in distributions as $\varepsilon \to 0$.
To simplify notations it is often convenient to replace (a) by

$$\text{(a')}\quad \mathrm{supp}(\rho_\varepsilon) \subseteq [-\varepsilon, \varepsilon] \quad \forall \varepsilon \in (0, 1).$$

This motivates the following (cf. [17])

[1] Note that since $\mathcal{D}$ is dense in $L^1$ practically even discontinuous regularizations (eg. boxes) are included.

**Definition 1** *A* generalized delta function *is an element $D$ of $\mathcal{G}(\mathbb{R}^n)$ possessing a representative $(D(\varphi,\,.\,))_{\varphi\in\mathcal{A}_0}$ such that $\exists p\in\mathbb{N}_0\ \forall\varphi\in\mathcal{A}_p(\mathbb{R}^n)\ \exists\eta=\eta(\varphi)>0$ :*

$$\begin{array}{ll}\text{(i)} & \mathrm{supp}(D(\varphi_\varepsilon,\,.\,))\subseteq[-\varepsilon,\varepsilon]\ \ \forall\varepsilon\in(0,\eta)\\ \text{(ii)} & \displaystyle\int D(\varphi_\varepsilon,x)\,dx\to 1\ (\varepsilon\to 0)\\ \text{(iii)} & \exists C=C(\varphi)>0 \textit{ such that } \displaystyle\int|D(\varphi_\varepsilon,x)|\,dx\le C\ \ \forall\varepsilon\in(0,\eta)\end{array}$$

The canonical embedding $R=\iota(\delta)$ of course falls into this class but clearly there are many generalized delta functions that do not correspond to any distribution via $\iota$. Moreover, every generalized delta function is associated to $\delta$, i.e. all generalized delta functions equal $\delta$ on the distributional level. In a sense, they may be viewed as 'delta distributions with a more refined microstructure' (fixing the additional nonlinear properties of the singularity).
Again, condition (i) in definition 1 has been chosen in order to avoid technicalities in the proofs of the following results, which, however, remain true if (i) is replaced by

$$\text{(i')}\quad \mathrm{supp}(R(\varphi_\varepsilon,\,.\,))\to\{0\}\qquad(\varepsilon\to 0)$$

Finally, we need the following technical preparation (which is actually a generalization of appendix A of [12]).

**Lemma 1** *Let $g:\mathbb{R}^n\to\mathbb{R}^n$, $h:\mathbb{R}\to\mathbb{R}^n$ smooth and let $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ be a net of smooth functions satisfying (a') and (c). For any $x_0,\dot{x}_0\in\mathbb{R}^n$ and any $\varepsilon\in(0,1)$ consider the system*

$$\begin{aligned}\ddot{x}_\varepsilon(t)&=g(x_\varepsilon(t))\rho_\varepsilon(t)+h(t)\\ x_\varepsilon(-1)&=x_0\\ \dot{x}_\varepsilon(-1)&=\dot{x}_0\end{aligned}\tag{3}$$

*Let $b>0$, $M=\int_{-1}^1\int_{-1}^s|h(r)|\,drds$, $I=\{x\in\mathbb{R}^n:\ |x-x_0|\le b+|\dot{x}_0|+M\}$ and $\alpha=\min\left\{\frac{b}{C\|g\|_{L^\infty(I)}+|\dot{x}_0|},\frac{1}{2LC},1\right\}$ with $L$ a Lipschitz constant for $g$ on $I$. Then (3) has a unique solution on $J_\varepsilon=[-1,\alpha-\varepsilon]$. Consequently, for $\varepsilon$ sufficiently small $x_\varepsilon$ is globally defined and both $x_\varepsilon$ and $\dot{x}_\varepsilon$ are bounded, uniformly in $\varepsilon$, on compact sets.*

**Proof.** The operator $f\to Af$,

$$Af(t)=x_0+\dot{x}_0(t+1)+\int_{-1}^{t}\int_{-1}^{s}g(f(r))\rho_\varepsilon(r)\,drds+\int_{-1}^{t}\int_{-1}^{s}h(r)\,drds$$

is a contraction on the complete metric space

$$\{f\in C(J_\varepsilon,\mathbb{R}^n):\ |f(t)-x_0|\le b+M+|\dot{x}_0|\}$$

□

Let us now turn to the geodesic equation for the pp-wave metric (2). Using $u$ as an affine parameter (which excludes trivial geodesics parallel to the shock) we obtain (cf. [12]):

$$\begin{aligned}\ddot{v}(u)&=f(x^i(u))\,\dot{\delta}(u)\ +\ 2\,\partial_i\,f(x^i(u))\,\dot{x}^i\,(u)\,\delta(u)\\ \ddot{x}^i(u)&=\frac{1}{2}\,\partial_i\,f(x^i(u))\,\delta(u)\end{aligned}\tag{4}$$

Since all operations appearing in (4) are well-defined in $\mathcal{G}$ (cf. the remarks following Theorem 1 below), we may seek solutions of the corresponding initial value problem in the Colombeau algebra by embedding $\delta(u)$ into $\mathcal{G}$. In fact, it turns out that for *any* generalized delta function there exists a unique solution. Denoting the generalized functions corresponding to $x^i$ and $v$ by capital letters we state the following

**Theorem 1** *Let $D\in\mathcal{G}(\mathbb{R})$ be a generalized delta function, $f\in\mathcal{C}^\infty(\mathbb{R}^2)$ and let $v_0,\ \dot{v}_0,\ x_0^i,\ \dot{x}_0^i\in\mathbb{R}$ $(i=1,2)$. The initial value problem*

$$
\begin{aligned}
\ddot{V}(u) &= f(X^i(u))\,\dot{D}(u) \,+\, 2\,\partial_i f(X^i(u))\,\dot{X}^i\,(u)\,D(u) \\
\ddot{X}^i(u) &= \frac{1}{2}\,\partial_i f(X^i(u))\,D(u) \\
V(-1) = v_0 &\qquad X^i(-1) \;=\; x_0^i \\
\dot{V}(-1) = \dot{v}_0 &\qquad \dot{X}^i(-1) \;=\; \dot{x}_0^i
\end{aligned}
\tag{5}
$$

*has a unique locally bounded solution* $(V, X^1, X^2) \in \mathcal{G}(\mathbb{R})^3$.

Note that we impose initial conditions in $u = -1$, i.e. "long before" the shock. Choosing initial conditions at $u = 0$ would mean to start "at the shock," which inevitably leads to regularization dependent weak limits.

**Proof.** *Existence:* Choose $p \in \mathbb{N}$ as in definition 1, fix $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ and let $\varepsilon < \eta(\varphi)$. Then componentwise we obtain the equations

$$
\begin{aligned}
\ddot{V}(\varphi_\varepsilon, u) &= f(X^i(\varphi_\varepsilon, u))\dot{D}(\varphi_\varepsilon, u) \,+\, 2\,\partial_i f(X^i(\varphi_\varepsilon, u))\dot{X}^i\,(\varphi_\varepsilon, u)\,D(\varphi_\varepsilon, u) \\
\ddot{X}^i(\varphi_\varepsilon, u) &= \frac{1}{2}\partial_i f(X^i(\varphi_\varepsilon, u))\,D(\varphi_\varepsilon, u) \\
V(\varphi_\varepsilon, -1) = v_0 &\qquad X^i(\varphi_\varepsilon, -1) = x_0^i \\
\dot{V}(\varphi_\varepsilon, -1) = \dot{v}_0 &\qquad \dot{X}^i(\varphi_\varepsilon, -1) = x_0^i
\end{aligned}
\tag{6}
$$

According to Lemma 1, the second line of (6) has a unique globally defined solution $X^i(\varphi_\varepsilon,\,.\,)$ with the specified initial values. Inserting this into the first line and integrating we also obtain a solution $V(\varphi_\varepsilon,\,.\,)$. From the boundedness properties of $X^i(\varphi_\varepsilon,\,.\,)$ established in Lemma 1 and the fact that $(D(\varphi,\,.\,))_{\varphi\in\mathcal{A}_0} \in \mathcal{E}_M(\mathbb{R})$ it follows easily by induction that $(X^i(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}$ and $(V(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}$ are moderate as well. Hence their respective classes in $\mathcal{G}(\mathbb{R})$ define solutions to (5).

*Uniqueness:* Suppose that $V_1 = \mathrm{cl}[(V_1(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}]$ and $X_1^i = \mathrm{cl}[(X^i(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}]$ are locally bounded solutions of (5) as well. On the level of representatives this means that there exist $M = \mathrm{cl}[(M(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}]$, $N^i = \mathrm{cl}[(N^i(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}] \in \mathcal{N}(\mathbb{R})$ and $n_{x^i},\, n_{\dot{x}^i},\, n_v,\, n_{\dot{v}} \in \mathcal{N}(1)$ with

$$
\begin{aligned}
\ddot{V}_1(\varphi_\varepsilon, u) \;\;&= f(X_1^i(\varphi_\varepsilon, u))\,\dot{D}(\varphi_\varepsilon, u) \,+\, 2\,\partial_i f(X_1^i(\varphi_\varepsilon, u))\,\dot{X}_1^i\,(\varphi_\varepsilon, u) \\
&\quad D(\varphi_\varepsilon, u) + M(\varphi_\varepsilon, u) \\
\ddot{X}_1^i(\varphi_\varepsilon, u) \;\;&= \frac{1}{2}\partial_i f(X_1^i(\varphi_\varepsilon, u))D(\varphi_\varepsilon, u) + N^i(\varphi_\varepsilon, u) \\
V_1(\varphi_\varepsilon, -1) = v_0 + n_v(\varphi_\varepsilon) &\quad X_1^i(\varphi_\varepsilon, -1) = x_0^i + n_{x^i}(\varphi_\varepsilon) \\
\dot{V}_1(\varphi_\varepsilon, -1) = \dot{v}_0 + n_{\dot{v}}(\varphi_\varepsilon) &\quad \dot{X}_1^i(\varphi_\varepsilon, -1) = \dot{x}_0^i + n_{\dot{x}^i}(\varphi_\varepsilon)
\end{aligned}
\tag{7}
$$

We have to show that $((V - V_1)(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}$ and $((X^i - X_1^i)(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0}$ belong to the ideal $\mathcal{N}(\mathbb{R})$. Since $N^i \in \mathcal{N}(\mathbb{R})$ it follows that for $p$ sufficiently large, $\varepsilon$ small and $\varphi \in \mathcal{A}_p(\mathbb{R})$, $N^i(\varphi_\varepsilon,\,.\,)$ is bounded on compact sets, uniformly in $\varepsilon$. Thus by Lemma 1 the same holds true for $X_1^i(\varphi_\varepsilon,\,.\,)$ and its first derivative. From (7) we conclude

$$
\begin{aligned}
(X^i - X_1^i)(\varphi_\varepsilon, u) &= -n_{x^i}(\varphi_\varepsilon) - (u+1)n_{\dot{x}^i}(\varphi_\varepsilon) + \\
&+\frac{1}{2}\int_{-1}^{u}\int_{-1}^{s} D(\varphi_\varepsilon, r)[\partial_i f(X^i(\varphi_\varepsilon, r)) - \partial_i f(X_1^i(\varphi_\varepsilon, r))]drds - \int_{-1}^{u}\int_{-1}^{s} N(\varphi_\varepsilon, r)drds
\end{aligned}
$$

Hence $\forall T > 0\ \exists p \in \mathbb{N}_0\ \exists \gamma \in \Gamma\ \forall q \geq p\ \forall \varphi \in \mathcal{A}_q(\mathbb{R})\ \exists C > 0\ \exists \eta > 0\ \forall \varepsilon \in (0, \eta)\ \forall u \in [-T, T]$:

$$
\begin{aligned}
|(X^i - X_1^i)(\varphi_\varepsilon, u)| &\leq C\varepsilon^{\gamma(q)-p} + \frac{1}{2}\int_{-1}^{u}\int_{-r}^{u}\int_{0}^{1} |\nabla\partial_i f(\sigma X^i(\varphi_\varepsilon, r)) + \\
&\quad +(1-\sigma)X_1^i(\varphi_\varepsilon, r))|d\sigma|(X^i - X_1^i)(\varphi_\varepsilon, r)||D(\varphi_\varepsilon, r)|dsdr
\end{aligned}
\tag{8}
$$

By the boundedness properties of $X^i$ and $X_1^i$ and by (iii), an application of Gronwall's Lemma to the above inequality yields the $\mathcal{N}$-estimates of order 0 for $(X^i - X_1^i)$. A similar argument applies to the first derivatives. The estimates of higher order then follow inductively from the differential equation, so $((X^i - X_1^i)(\varphi_\varepsilon,\,.\,))_{\varphi\in\mathcal{A}_0} \in \mathcal{N}(\mathbb{R})$. Inserting this into the integral equation for $(V - V_1)$, the $\mathcal{N}$-estimates for $(V - V_1)$ also follow inductively. □

In the proof of Theorem 1 we have only made use of properties (i) and (iii) of the generalized delta function $D$. On the other hand, property (ii) will be essential for the explicit calculation of distributional limits of the unique solution constructed in Theorem 1, cf. Sec. IV. Also, note that we did not have to impose any growth restrictions on $f$ to obtain a well-defined element $f(X^i)$ of $\mathcal{G}$. This is of course due to the fact that any componentwise solution of the initial value problem necessarily is bounded, uniformly in $\varepsilon$, on compact sets (for $\varepsilon$ small).
Our next goal is an analysis of the Jacobi equation for impulsive pp-waves in the framework of algebras of generalized functions. As in [12] to keep formulas more transparent we make some simplifying assumptions concerning geometry (namely axisymmetry) and initial conditions. Writing $x = x^1$ and $y = x^2$ we suppose that $f$ depends exclusively on the two-radius $\sqrt{x^2 + y^2}$ and work within the hypersurface $y = 0$ (corresponding to initial conditions $y_0 = 0 = \dot{y}_0$). Furthermore we demand $v_0 = 0 = \dot{x}_0$. As was shown in [12], in this situation the Jacobi equation

$$\frac{D^2 N^a}{dt^2} = -R^{\,a}_{\,bcd} T^b T^d N^c \ ,$$

where $N^a(u) = (N^u(u), N^v(u), N^x(u), N^y(u))$ denotes the deviation vector field takes the form

$$\begin{aligned}
\ddot{N}^v &= 2[N^x f'(x)\delta]^{\cdot} - N^x f'(x)\dot{\delta} + [N^u f(x)\delta]^{\cdot\cdot} - N^u f''(x)\dot{x}^2\delta - N^u f'(x)\ddot{x}\delta \\
\ddot{N}^x &= [\dot{N}^u f'(x) + \frac{1}{2} N^x f''(x)]\delta + \frac{1}{2} f'(x) N^u \dot{\delta} \\
\ddot{N}^y &= \ddot{N}^u \ = \ 0 \ ,
\end{aligned} \tag{9}$$

where $x$ is determined by (4). Existence and uniqueness of solutions to the corresponding initial value problem in the Colombeau algebra is established in the following result where, for the sake of brevity we denote the $\mathcal{G}$-functions corresponding to $N^a$ again by $N^a$.

**Theorem 2** *Let $D \in \mathcal{G}(\mathbb{R})$ be a generalized delta function, $f \in \mathcal{C}^\infty(\mathbb{R})$, $n^a$, $\dot{n}^a \in \mathbb{R}^4$ and let $X$ denote the (unique) solution to system (5) with initial conditions and simplifications as discussed above. The initial value problem*

$$\begin{aligned}
\ddot{N}^v &= 2[N^x f'(X)D]^{\cdot} - N^x f'(X)\dot{D} + [N^u f(X)D]^{\cdot\cdot} - \\
&\quad - N^u f''(X)\dot{X}^2 D - N^u f'(X)\ddot{X} D \\
\ddot{N}^x &= [\dot{N}^u f'(X) + \frac{1}{2} N^x f''(X)]D + \frac{1}{2} f'(X) N^u \dot{D} \\
\ddot{N}^y &= \ddot{N}^u \ = \ 0 \\
N^a(-1) &= n^a \qquad \dot{N}^a(-1) \ = \ \dot{n}^a
\end{aligned} \tag{10}$$

*has a unique solution $N^a \in \mathcal{G}(\mathbb{R})^4$.*

**Proof.** Since the equations are linear in the components of the deviation field we are provided with globally defined solutions on the level of representatives. The last two equations are actually trivial and so is the first one once we know that its right hand side belongs to $\mathcal{G}(\mathbb{R})$. Hence we are left with the equation for $N^x$ which is of the form $\ddot{N}(t) = f''(X(t))D(t)N(t) + H(t)$ with $H$ in $\mathcal{G}(\mathbb{R})$. Using the boundedness properties of $X$ established in Lemma 1 the $\mathcal{E}_M$-bounds for $N^x$ easily follow from Gronwall's lemma.
Uniqueness is established along the same lines again using Gronwall-type arguments. □
In the above proof we have again only used properties (i) and (iii) of the generalized delta function $D$.
To conclude this section we remark that unique solvability of the geodesic and geodesic deviation equation for (2) is not confined to the case where the profile function $f$ is smooth. Indeed, it turns out that for a large class of generalized profile functions (those that are not "too singular") Theorems 1 and 2 retain their validity. More precisely, we have to demand that $f$ belongs to the algebra of *tempered* generalized functions [15] to make sure that the composition $f(X)$ is well defined and that $\nabla\nabla f$ is of $L^\infty$-log-type [17,18] to ensure existence and uniqueness of solutions to (5) and (10). However, to include many physically interesting examples (cf. [21]) one has to cut out the worldline of the ultrarelativistic particle, i.e. the $v$-axis from the domain of definition (cf. [22]).

## IV. DISTRIBUTIONAL LIMITS

In this section we are going to calculate the distributional limits (or, in the terminology of Colombeau theory: the associated distributions) of the unique solutions to the geodesic and geodesic deviation equation constructed in

Theorems 1 and 2. In [12] distributional limits for regularized versions of these equations have been calculated using a model delta net regularization.Translated into our current setting this amounts to using the particular generalized delta function $D = \iota(\delta)$. Our aim is to extend the validity of the limit relations derived there to the case of solutions in the Colombeau algebra and to generalized delta functions. At the same time we will be able to prove stronger convergence results in some cases.

**Theorem 3** *The unique solution $(V, X^i)$ of the geodesic equation (5) satisfies the following association relations:*

$$X^i \approx x_0^i + \dot{x}_0^i(1+u) + \frac{1}{2}\partial_i f(x_0^i + \dot{x}_0^i)u_+ \tag{11}$$

$$V \approx v_0 + \dot{v}_0(1+u) + f(x_0^i + \dot{x}_0^i)\theta(u) + \partial_i f(x_0^i + \dot{x}_0^i)\left(\dot{x}_0^i + \frac{1}{4}\partial_i f(x_0^i + \dot{x}_0^i)\right)u_+ \tag{12}$$

*In addition, if $X^i = \mathrm{cl}[(X^i(\varphi, \,.\,))_{\varphi\in\mathcal{A}_0}]$ then $\exists p \in \mathbb{N}_0$ such that $\forall \varphi \in \mathcal{A}_p$*

$$X^i(\varphi_\varepsilon, u) \to x_0^i + \dot{x}_0^i(1+u) + \frac{1}{2}\partial_i f(x_0^i + \dot{x}_0^i)u_+ \tag{13}$$

*for $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}$.*

**Proof.** Choose $p \in \mathbb{N}_0$ as in definition 1 for $D$, let $\varphi \in \mathcal{A}_p$ and $\varepsilon < \eta(\varphi)$. Since integrating amounts to convolution with the Heaviside function, which is a continuous operation on the convolution algebra of distributions supported in a cone, in order to prove (11) it suffices to show that

$$\ddot{X}^i(\varphi_\varepsilon, \,.\,) = \frac{1}{2}\partial_i f(X^i(\varphi_\varepsilon, \,.\,))D(\varphi_\varepsilon, \,.\,) \to \frac{1}{2}\partial_i f(x_0^i + \dot{x}_0^i)\delta$$

in distributions. We first note that $X(\varphi_\varepsilon, \varepsilon t) \to x_0^i + \dot{x}_0^i$ uniformly as can be seen from the integral equation for $X^i$ (cf. (11) in [12]). Now if $\psi \in \mathcal{D}(\mathbb{R})$ then

$$\begin{aligned}
&\Big| \int_{-\epsilon}^{\epsilon} \psi(t)\partial_i f(X^i(\varphi_\epsilon, t))D(\varphi_\epsilon, t)dt - \partial_i f(x_0^i + \dot{x}_0^i)\psi(0) \Big| \\
&\le \sup_{-\epsilon\le t\le\epsilon} \mid \psi(t)\partial_i f(X^i(\varphi_\epsilon, t)) - \partial_i f(x_0^i + \dot{x}_0^i)\psi(0) \mid \int_{-\epsilon}^{\epsilon} \mid D(\varphi_\epsilon, t) \mid dt + \\
&+ \int_{-\epsilon}^{\epsilon} \mid D(\varphi_\epsilon, t)dt - 1 \mid \partial_i f(x_0^i + \dot{x}_0^i)\psi(0)
\end{aligned}$$

So the claim follows from properties (iii) and (ii) of the generalized delta function $D$. Since $\dot{X}^i(\varphi_\varepsilon, t)$ is bounded on compact sets, uniformly in $\varepsilon$, it follows that the family $\{X^i(\varphi_\varepsilon, t) : \varepsilon \in (0,1)\}$ is locally equicontinuous. Hence Ascoli's Theorem implies (13). Concerning (12), as above it suffices to calculate the limit of

$$\ddot{V}(\varphi_\varepsilon, u) = [f(X^i(\varphi_\varepsilon, u))D(\varphi_\varepsilon, u)]^{\cdot} + \partial_i f(X^i(\varphi_\varepsilon, u))\dot{X}^i(\varphi_\varepsilon, u)D(\varphi_\varepsilon, u)$$

whose first summand converges to $f(x_0^i + \dot{x}_0^i)\dot{\delta}$ by an argument similar to the one above. For the second summand we have

$$\begin{aligned}
\partial_i f(X^i(\varphi_\varepsilon, u))\dot{X}^i(\varphi_\varepsilon, u)D(\varphi_\varepsilon, u) &= \underbrace{\partial_i f(X^i(\varphi_\varepsilon, u))D(\varphi_\varepsilon, u)\dot{x}_0^i}_{(*)} \\
&+\frac{1}{2}\partial_i f(X^i(\varphi_\varepsilon, u))D(\varphi_\varepsilon, u)\int_{-\varepsilon}^{t} \partial_i f(X^i(\varphi_\varepsilon, s))D(\varphi_\varepsilon, s)ds
\end{aligned}$$

and $(*) \to \partial_i f(x_0^i + \dot{x}_0^i)\dot{x}_0^i\delta$. Finally, since

$$\int_{-\varepsilon}^{\varepsilon} \psi(t)\partial_i f(X^i(\varphi_\varepsilon, t))D(\varphi_\varepsilon, t) \int_{-\varepsilon}^{t} \partial_i f(X^i(\varphi_\varepsilon, s))D(\varphi_\varepsilon, s)dsdt -$$

$$-\frac{1}{2}\partial_i f(x_0^i + \dot{x}_0^i)^2\psi(0) \int_{-\varepsilon}^{\varepsilon} D(\varphi_\varepsilon, t)dt \to 0\,,$$

the claim follows. □

In calculating distributional limits for the solution of the Jacobi equation to maintain clarity of formulae we shall make simplifying assumptions on the initial conditions, i.e.

$$\begin{aligned} N^a(-1) &= (0,0,0,0) \\ \dot{N}^a(-1) &= (a,b,0,0)\ . \end{aligned} \tag{14}$$

Then we have

**Theorem 4** *The unique solution of the geodesic deviation equation (10) satisfies the following association relations:*

$$N^x \approx \frac{1}{2} a f'(x_0)(u_+ + \theta(u)) \tag{15}$$

$$N^v \approx b(1+u) + a[f(x_0)\delta(u) + \frac{1}{4} f'(x_0)^2(\theta(u) + u_+)] \tag{16}$$

**Proof.** The general structure of this proof is 'isomorphic' to the calculation of distributional limits for the regularized Jacobi equation in [12]. The main difference is that for representatives of generalized delta functions dominated convergence arguments are not applicable which makes the calculations more tedious. Nevertheless, using the uniform convergence of $X^i(\varphi_\epsilon, .)$ established above, all steps carried out in [12] can be adapted to the present situation as demonstrated in the proof of Theorem 3. □

## V. DISCUSSION AND OUTLOOK

In the previous section we have shown that the unique solutions to the geodesic and geodesic deviation equation in the Colombeau algebra possess a physically reasonable macroscopic (i.e. distributional) aspect: even within the natural maximal class of delta-regularizations (namely the class of all generalized delta functions) the regularity of the equations is sufficiently high to ensure distributional limits corresponding to physical expectations. More precisely, from the distributional point of view, the geodesics correspond to refracted, broken straight lines as suggested by the form of the metric. The scale of the jump and kink is given by the values of $f$ and its first derivatives at the shock hypersurface, which can be traced back to the values at the initial point ($u = -1$), thereby precisely reproducing Penrose's junction conditions [1]. The distributional limit of the Jacobi field suffers a kink and jump in $x$-direction as well as an additional $\delta$-pulse in $v$-direction, which may be understood from the form of the geodesics. For a more detailed discussion see [12].

Finally we make some comments on diffeomorphism invariance of our results. Whereas the fine sheaf of Colombeau algebras can be lifted to manifolds in a straightforward manner, the action of a diffeomorphism does not commute with the canonical embedding $\mathcal{D}' \hookrightarrow \mathcal{G}$. The reason for this is that convolution relies on the additive group structure of $\mathbb{R}^n$ and is therefore not invariant under the action of diffeomorphisms. Note, however that our calculations did not use the embedding via convolution and therefore are not affected by this defect.

A solution to the above mentioned problem was proposed in [23] using a modified definition of the mollifier spaces $\mathcal{A}_q$. A key ingredient of this construction is that diffeomorphisms act on the $\varphi$'s, introducing an implicit $x$-dependence into the first slot of the Colombeau functions $R(\varphi, x)$. Hence, to retain smooth $x$-dependence of $R$ the concept of Silva-differentiability was used. Future work will be concerned with a simplified concept of Colombeau algebras on manifolds using calculus in convenient vector spaces [24]. A main goal of this line of research is to provide a workable solution concept for singular differential equations on manifolds.

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